题：use the method of variation of parameters to find a particular solution for the equation $y''+16y=4\tan(4x)$.

解：$y''+16y=0$ 对应特征方程 $r^2+16=0$，$r=\pm 4i$

则齐次通解为 $y=C_1\cos(4x)+C_2\sin(4x)$

设原非齐次方程的特解为：$y_p(x)=u(x)\cos 4x+v(x)\sin(4x)$

$$y_p'(x)=u'(x)\cos 4x+u(x)(-\sin 4x)\cdot 4+v'(x)\sin 4x+v(x)\cos 4x\cdot 4$$
$$=u'(x)\cos 4x+v'(x)\sin 4x-4u(x)\sin 4x+4v(x)\cos 4x$$

$u'(x)\cos 4x+v'(x)\sin 4x=0$ ①

$y_p'=-4u(x)\sin 4x+4v(x)\cos 4x$

$$y_p''=-4u'(x)\sin 4x-4u(x)\cos 4x\cdot 4+4v'(x)\cos 4x+4v(x)(-\sin 4x)\cdot 4$$
$$=-4u'(x)\sin 4x+4v'(x)\cos 4x-16u(x)\cos 4x-16v(x)\sin 4x$$

把 $y_p''$，$y_p$ 代入方程 $y''+16y=4\tan 4x$ 得：

$-4u'(x)\sin 4x+4v'(x)\cos 4x=4\tan 4x$ ②

①，② 组成方程组 $\begin{cases} u'(x)\cos 4x+v'(x)\sin 4x=0 \\ u'(x)(-4\sin 4x)+v'(x)4\cos 4x=4\tan 4x \end{cases}$

$A=\begin{pmatrix}\cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x\end{pmatrix}$，$\det A=4\cos^2 4x+4\sin^2 4x=4\neq 0$

$$u'(x)=\frac{\det\begin{pmatrix}0 & \sin 4x \\ 4\tan 4x & 4\cos 4x\end{pmatrix}}{\det A}=\frac{-4\tan 4x\sin 4x}{4}=-\tan 4x\sin 4x$$

$$v'(x)=\frac{\det\begin{pmatrix}\cos 4x & 0 \\ -4\sin 4x & 4\tan 4x\end{pmatrix}}{\det A}=\frac{\cos 4x\, 4\tan 4x}{4}=\sin 4x$$

$$u(x)=\int -\tan 4x\sin 4x\,dx=-\int\frac{\sin^2 4x}{\cos 4x}dx=-\int\frac{1}{\cos 4x}-\cos 4x\,dx$$
$$=-\int\sec 4x\,dx+\frac{1}{4}\sin 4x=-\frac{1}{4}\ln|\sec 4x+\tan 4x|+\frac{1}{4}\sin 4x$$

$v(x)=\int\sin 4x\,dx=-\frac{1}{4}\cos 4x$

$$y_p=\left(\frac{1}{4}\sin 4x-\frac{1}{4}\ln|\sec 4x+\tan 4x|\right)\cos 4x-\frac{1}{4}\cos 4x\sin 4x$$
$$=-\frac{1}{4}\cos 4x\ln|\sec 4x+\tan 4x|$$ 解毕.