题：Show that for $n \geq 2$

$$\frac{1}{n}\sum_{k=2}^{n} k\binom{n}{k}3^{k-1} = (2^{n-1}+1)\sum_{k=0}^{n-2}2^k$$

证：let $f(x)=(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k = 1+\sum_{k=1}^{n}\binom{n}{k}x^k$

$$f'(x)=n(1+x)^{n-1}=\sum_{k=1}^{n}k\binom{n}{k}x^{k-1}=n+\sum_{k=2}^{n}k\binom{n}{k}x^{k-2}$$

$$f'(3)=n\cdot 4^{n-1}=n+\sum_{k=2}^{n}k\binom{n}{k}3^{k-2}$$

$$4^{n-1}=1+\frac{1}{n}\sum_{k=2}^{n}k\binom{n}{k}3^{k-2}$$

$$\frac{1}{n}\sum_{k=2}^{n}k\binom{n}{k}3^{k-2}=4^{n-1}-1=2^{2n-2}-1$$

$$(2^{n-1}+1)\sum_{k=0}^{n-2}2^k=(2^{n-1}+1)(2^0+2^1+2^2+\cdots+2^{n-2})$$

$$=(2^{n-1}+1)\left(\frac{2^0(1-2^{n-1})}{1-2}\right)=(2^{n-1}+1)(2^{n-1}-1)$$

$$=(2^{n-1})^2-1=2^{2n-2}-1$$

$$\therefore \frac{1}{n}\sum_{k=2}^{n}k\binom{n}{k}3^{k-1}=(2^{n-1}+1)\sum_{k=0}^{n-2}2^k$$