

Problem: Find the Jordan form for each of the matrices  $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . Are A and B similar?

Explain your answers.

Solution:  $\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{bmatrix} = \lambda^4 = 0, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0. \lambda = 0, (A - \lambda I)X = 0,$

$X = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} C_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} C_2,$  there is only two linear independent eigenvectors for four multiple root  $\lambda = 0$ . The Jordan Form

for A is  $J_A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$  So there is invertible matrix P,  $P^{-1}AP = J_A, AP = PJ_A,$

let  $P = [V_1 \ V_2 \ V_3 \ V_4], V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, AV_2 = V_1, AV_4 = V_3, \text{ So } V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, V_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Similarly,  $\det(B - \lambda I) = \lambda^4 = 0, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0,$  there is only two linear independent eigenvector for four multiple root

$\lambda = 0.$  The Jordan Form for B is  $J_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  So there is invertible matrix T,

$T^{-1}BT = J_B, BT = TJ_B,$  let  $T = [u_1 \ u_2 \ u_3 \ u_4], u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, Bu_2 = u_1, Bu_3 = u_2, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Assume that A is similar to B, So there is invertible matrix X,  $X^{-1}AX = B, AX = XB,$  let  $X = [x_1 \ x_2 \ x_3 \ x_4],$

$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, Ax_2 = x_1, x_2 = \begin{bmatrix} 1 \\ 0 \\ * \\ * \end{bmatrix}, Ax_3 = x_2 \text{ havn't solution, if } X^{-1}AX = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$

$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$   $x_3$  and  $x_4$  are not exist. So the assumption is false, this is, matrix A is not similar to B.