题关键词：换元积分法，定积分性质，定积分换元

题：a) 利用换元积分法证明

$$\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)dx$$

a)并利用上式求

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

解：a) 令 x=π-t, 则 dx= - dt, sinx=sint,

$$\int_0^{\pi} xf(\sin x)dx = -\int_{\pi}^{0}(\pi-t)f(\sin t)dt = \int_0^{\pi}(\pi-t)f(\sin t)dx = \int_0^{\pi}\pi f(\sin x)dx - \int_0^{\pi} tf(\sin t)dx$$

$\because \int_0^{\pi} xf(\sin x)dx$等同于$\int_0^{\pi} xf(\sin x)dx$，$\therefore 2\int_0^{\pi} xf(\sin x)dx = \int_0^{\pi}\pi f(\sin x)dx$

令 x=π-t, 则 dx= - dt, sinx=sint , $\int_{\frac{\pi}{2}}^{\pi} f(\sin x)dx = -\int_{\frac{\pi}{2}}^{0} f(\sin t)dt = \int_0^{\frac{\pi}{2}} f(\sin t)dtx$

所以$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x)dx + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi} f(\sin x)dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x)dx$

$+\frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x) = \pi\int_0^{\frac{\pi}{2}} f(\sin x)$，证毕。

b) 令 u=cosx， du= - sinxdx，x=π/2, u=0, x=0, u=1

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = \pi\int_1^0 \frac{-1}{1+u^2}du = \left[-\pi\arctan u\right]_1^0 = \frac{\pi^2}{4}$$