题：Let S∈L(v,v) be given by $S(u_1)=u_1-u_2$, $S(u_2)=u_1$, where $\{u_1,u_2\}$ is a basis for V. Find the matrices of S with respect to the basis $\{u_1,u_2\}$ and with respect to the new basis $\{w_1,w_2\}$ where $w_1=3u_1-u_2$, $w_2=u_1+u_2$. Find invertible matrices X in each case such that $X^{-1}AX=B$ where A is the matrix of the transformation with respect to the old basis, and B is the matrix with respect to the new basis.

Solution:

Since $\begin{cases} S(u_1)=u_1-u_2 \\ S(u_2)=u_1 \end{cases}$, so S is linear transformation $S=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. $\begin{cases} w_1=3u_1-u_2 \\ w_2=u_1+u_2 \end{cases}$,

$\begin{cases} u_1=\frac{1}{4}w_1+\frac{1}{4}w_2 \\ u_2=-\frac{1}{4}w_1+\frac{3}{4}w_2 \end{cases}$, $S(w_1)=S(3u_1-u_2)=3S(u_1)-S(u_2)=3(u_1-u_2)-u_1=2u_1-3u_2=$

$2\left(\frac{1}{4}w_1+\frac{1}{4}w_2\right)-3\left(-\frac{1}{4}w_1+\frac{3}{4}w_2\right)=\frac{5}{4}w_1-\frac{7}{4}w_2$,

$S(w_2)=S(u_1+u_2)=2S(u_1)+2S(u_2)=u_1-u_2+u_1=2u_1-u_2=2(\frac{1}{4}w_1+\frac{1}{4}w_2)-(-\frac{1}{4}w_1+\frac{3}{4}w_2)=\frac{3}{4}w_1-\frac{1}{4}w_2$

$\begin{cases} S(w_1)=\frac{5}{4}w_1-\frac{7}{4}w_2 \\ S(w_2)=\frac{3}{4}w_1-\frac{1}{4}w_2 \end{cases}$, so, the matrices of S with respect to the basis $\{w_1,w_2\}$ is

$S_w=\begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ -\frac{7}{4} & -\frac{1}{4} \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 5 & 3 \\ -7 & -1 \end{pmatrix}$.

$\begin{cases} 3u_1-u_2=w_1 \\ u_1+u_2=w_2 \end{cases}$ let X is the matrix of the linear transformation from the basis $\{u_1,u_2\}$ to the new basis $\{w_1,w_2\}$, $X=\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$, $X^{-1}=\frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

$X^{-1}SX=\frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 5 & 3 \\ -7 & -1 \end{pmatrix}=S_w$.