题关键词：eigenvalues and eigenvector, diagonalization, inner product, invertible matrix.
In this question A is the matrix $A:=UV^T$, where u and v are non-zero vector in R^{10}.

1. If u·v≠0 find all 10 eigenvalues and describe their corresponding eigenvectors
   a) Decide if A is diagonalizeble.
   b) Decide if A is invertible.
2. If u·v≠0 find all 10 eigenvalues and describe their corresponding eigenvectors
   c) Decide if A is diagonalizable.
   d) Decide if A is invertible.

Solution:

Let $U=(a_1,a_2,...,a_{10})^T$, $V=(b_1,b_2,...,b_{10})^T$.

so $A=UV^T=(a_1,a_2,...,a_{10})^T(b_1,b_2,...,b_{10})=\begin{bmatrix} a_1b_1 & a_1b_2 & ... & a_1b_{10} \\ a_2b_1 & a_2b_2 & ... & a_2b_{10} \\ .. & .. & .. & .. \\ a_{10}b_1 & a_{10}b_2 & ... & a_{10}b_{10} \end{bmatrix} \xrightarrow{\text{经初等变换}} \begin{bmatrix} b_1 & b_2 & ... & b_{10} \\ b_1 & b_2 & ... & b_{10} \\ .. & .. & .. & .. \\ b_1 & b_2 & ... & b_{10} \end{bmatrix}$.

then Rank(A)=1.

Let X is an eigenvector belonging to the eigenvalue λ.

so $AX=\lambda X$, $UV^TX=\lambda X$, $UV^TUV^TX=UV^T\lambda X=\lambda UV^TX=\lambda^2X$, so $(V^TU)UV^TX=\lambda^2X$

1) a)Since u·v≠0, then $V^TU\neq0$, so $UV^TX=\frac{\lambda^2}{V^TU}X=\lambda X$, $\frac{\lambda^2}{V^TU}=\lambda$，解得$\lambda=0$ *or* $\lambda=1$

When λ=0, $UV^TX=O$, $Rank(UV^T)=1$. $UV^TX=O$ have n-1 linearly independent solution, that is,λ=0 corresponding to n-1 linearly independent eigenvectors, so λ=0 is an eigenvalue of multiplicity n-1, so A has n linearly independent eigenvectors, then A is diaonalizable. The eigenspace of A corresponding to λ=0 is null(A) and the eigenspace of A corresponding to λ=1 is $null(UV^T-I)$

b). $Rank(A_{10X10})=1<10$, det(A)=0, thus A is not invertible.

2) a)when u·v=0, $V^TU=0$, so $0=\lambda^2X$, then λ=0 is an eigenvalue of multiplicity n, and corresponding to n-1 linearly independent eigenvectors, thus A is not diaonalizable.

b). Same to 1) $Rank(A_{10X10})=1<10$, det(A)=0, thus A is not invertible.